SOME SOLUTIONS OF THE NONSTATIONARY HEAT-CONDUCTION EQUATION IN A REGIME WITH SLIPPAGE

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We present an exact solution of the nonstationary heat-conduction equation for particles belonging to a nonhomogeneously heated gas with a temperature field linear at infinity and with all corrections, linear with respect to the Knudsen number, taken into account.

For small nonzero Knudsen numbers Kn the ordinary hydrodynamics equations, including the heat-conduction equation, become inaccurate close to a solid surface in a kinetic layer on the order of several free path lengths of the gas particles. However, within the framework of ordinary hydrodynamics, one can find first-order corrections to the hydrodynamic equations, which are linear in the Knudsen number, with the aid of which one may employ the corresponding boundary conditions on the solid body surface [1-4]. Constants appearing in the boundary conditions may be obtained from a comparison of the predicted theory with experiment [5] of from a solution of the kinetic equation close to a particle [3].

We consider a spherical particle situated in a temperature field linear at infinity. We shall assume that $\mathrm{Kn} = l/R \leqq 0.3$, $R|\nabla \ln T| \ll 1$. Under these conditions we can neglect the influence of slow thermophoretic motion on heat transfer and we can use the ordinary heat-conduction equation

$$\operatorname{div}(\varkappa \operatorname{grad} T) = c_p\rho \frac{\partial T}{\partial r} + F(\mathbf{r},\ t),$$
$$\varkappa = \begin{cases} \varkappa_a & r<R, \\ \varkappa_e & r>R, \end{cases} \qquad c_p\rho = \begin{cases} c_{pa}\rho_a & r<R, \\ c_{pe}\rho_e & r>R. \end{cases} \tag{1}$$

The initial and boundary conditions on the particle surface [3, 4] and at infinity have the form

$$T_e - T_a = K_\tau l\left(\frac{\partial T_e}{\partial r}\right)_R;\quad \varkappa_e\frac{\partial T_e}{\partial r} - \varkappa_a\frac{\partial T_a}{\partial r} = c_q \mathrm{Kn}\frac{\varkappa_e}{R}(\operatorname{div}\nabla_\tau T_e)_R;$$
$$T_{e_{r\to\infty}} \to T_{e0} + \mathbf{r}(\nabla T_e)_\infty;\quad T(t=0,\ \mathbf{r}) = q(\mathbf{r}), \tag{2}$$

where $\nabla_\tau T_e$ is the tengential portion of the temperature gradient on the particle surface. The first of the boundary conditions (2) takes into account the jump in temperature on the particle surface owing to the presence of the Knudsen layer, in which the ordinary equations do not apply. The second condition describes the tangential heat transfer (slippage) in the Knudsen layer, which has an affect on the balance of normal heat flows. The boundary conditions (2) enable us to "splice together" the solutions in the Knudsen layer where the Eq. (1) is not applicable.

We seek a solution in the form $T = T_1 + T_2$, where T_2 tends toward zero as $r \to \infty$. For T_1 we take the stationary solution. In a spherical coordinate system with z axis directed along the temperature gradient at infinity, T_1 has the form

$$T_{e1} = T_{e0} + \mathbf{r}(\nabla T_e)_\infty + \frac{B}{r^2}\cos\theta;\quad T_{a1} = T_{e0} + Ar\cos\theta;$$

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$$A = \frac{\varkappa_e + 2\alpha_1\alpha_2}{\varkappa_a + 2\varkappa_e - 2R\alpha_2 + \dfrac{2\varkappa_a\alpha_1}{R}}; \tag{3}$$

$$B = \frac{\varkappa_e - \varkappa_a + 2\alpha_2 R + \varkappa_a\alpha_1/R}{\varkappa_a + 2\varkappa_e - 2\alpha_2 R + 2\varkappa_a\alpha_1/R}; \quad \alpha_1 = K_T l; \quad \alpha_2 = c_q \mathrm{Kn}\, \varkappa_e/R.$$

We expand T_2 in the spherical functions $Y_{mn}(\theta, \varphi)$ and substitute it into Eqs. (1) and (2). For the coefficients of the expansion we obtain the equation

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\varkappa\frac{\partial}{\partial r}T_{nm}\right)\right] - \frac{n(n+1)}{r^2}\varkappa T_{nm} = c_p\rho\left(\frac{\partial T_{nm}}{\partial t} + F_{nm}\right);$$
$$T_2 = \sum_{n,m} T_{nm}(r, t)\, Y_{nm}(\theta, \varphi); \quad F = c_p\rho \sum_{n,m} F_{nm}(r, t)\, Y_{nm}(\theta, \varphi) \tag{4}$$

with the boundary conditions

$$[(T_e)_{nm} - (T_a)_{nm}]_{r=R} = \alpha_1\left[\frac{\partial}{\partial r}(T_e)_{nm}\right]_{r=R};$$
$$\left[\varkappa_e\frac{\partial}{\partial r}(T_e)_{nm} - \varkappa_a\frac{\partial}{\partial r}(T_a)_{nm}\right]_{r=R} = a_n (T_e)_{nm}; \quad a_n = -n(n+1)\alpha_2. \tag{5}$$

In solving the system (4), (5) it is much more convenient to use an integral transform with respect to the variable r rather than the Laplace transform with respect to the time, customarily applied in such cases. To do this, we multiply Eq. (4) by $g(r)K_{nm}(r, s)$ and integrate from zero to infinity. Putting $g(r) = r^2\beta$, we obtain the following equation after an integration by parts:

$$\int_0^\infty \beta\left\{\frac{\partial}{\partial r}\left(r^2\varkappa\frac{\partial}{\partial r}K_{nm}\right) - n(n+1)K_{nm}\right\} T_{nm}dr + N(R) = \frac{\partial}{\partial t}T_{nms}(t) + F_{nms}. \tag{6}$$

Here

$$N = N_a - N_e; \quad N_\alpha = \beta_\alpha\varkappa_\alpha R^2 W[(K_\alpha)_{nm}, (T_\alpha)_{nms}] \ (\alpha = a, e),$$

where W is the Wronskian and the function K_{nm} must satisfy the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\varkappa\frac{\partial}{\partial r}K_{nm}\right) - \frac{n(n+1)}{r^2}K_{nm}\varkappa = -s^2 c_p\rho K_{nm}. \tag{7}$$

If in Eq. (5) we express the derivatives of the temperature in terms of the remaining functions and substitute into Eq. (6), we find, if the conditions

$$\frac{\beta_e}{\beta_a}(K_e)_{nm} - (K_a)_{nm} = \alpha_1\frac{\varkappa_a}{\varkappa_e}\frac{\partial}{\partial r}(K_a)_{nm};$$
$$\varkappa_e\frac{\beta_e}{\beta_a}\frac{\partial}{\partial r}(K_e)_{nm} - \varkappa_a\frac{\partial}{\partial r}(K_a)_{nm} = a_n (K_a)_{nm} \tag{8}$$

are satisfied for r = R, that N = 0.

For the operator in Eq. (7) to be self-adjoint in [0, ∞] it is necessary to choose the weight g(r) to be discontinuous:

$$g_a(r) = \beta_a r^2; \ g_e(r) = \beta_e r^2; \ \beta_e = \beta_a(1 - \alpha_1 a_n/\varkappa_e) \equiv \beta. \tag{9}$$

With no loss of generality we can put $\beta_a = 1$.

Solving Eq. (7) along with Eq. (8) and taking Eq. (9) into account, we obtain

$$(K_a)_{nm} = \frac{\varkappa_e\beta}{k_e R^2}A_{nm}(s)\, J_n(k_a, r); \tag{10}$$
$$(K_e)_{nm} = A_{nm}(s)\{a_{nm}(s)\, J_n(k_e r) + b_{nm}(s)\, \chi_n(k_e r)\},$$

where

$$a_{nm} = \varkappa_e k_e J_n(k_a, R)\chi_n'(k_eR) - \varkappa_a k_a J_n'(k_a, R)\chi_n(k_e, R) +$$
$$+ \alpha_1 \varkappa_a k_a k_e J_n'(k_aR)\chi_n'(k_eR) - a_n J_n(k_aR)\chi_n(k_eR);$$
$$b_{nm} = \varkappa_a k_a J_n(k_e, R) J_n'(k_aR) - \varkappa_e k_e J_n'(k_eR) J_n(k_aR) +$$
$$+ a_n J_n(k_aR) J_n(k_eR) - \alpha_1 \varkappa_a k_a J_n'(k_eR); \tag{11}$$
$$k^2 = \frac{s^2}{a^2} = \frac{c_p\rho}{\varkappa} s^2;\ f_n'(kR) = \frac{\partial}{\partial kR} f_n;\ f_n = J_n,\ \chi_n.$$

Here J_n and χ_n are spherical Bessel and Neumann functions. The constants A_{nm} are obtained from the normalization condition

$$A_{nm}(s) = \left[\frac{\pi}{2}\ \frac{\varkappa_e - \alpha_1 a_n}{a_e k_e^2}(a_{nm}^2(s) - b_{nm}^2(s))\right]^{-1/2}.$$

Since N = 0, Eq. (5) assumes the form

$$\frac{\partial}{\partial t} T_{nm}(s, t) + s^2 T_{nm}(s, t) + F_{nm}(s, t) = 0.$$

The spectrum of the problem is continuous. The inverse transform is

$$T = T_1 + \sum_{n,m} Y_{nm}(\theta, \varphi) \int_0^\infty K_{nm}(r, s) T_{nm}(s, t)\, ds. \tag{12}$$

From Eqs. (10)-(12) we find, after certain transformations,

$$T_a = T_{a1} + \frac{2}{\pi R^2} a_e \varkappa_e \sum_{n,m} Y_{nm}(\theta, \varphi)\left[1 - \frac{\alpha_1 a_n}{\varkappa_e}\right] \times$$
$$\times \int_0^\infty \frac{J_n(k_a r)}{a_{nm}^2(s) + b_{nm}^2(s)} \int_0^\infty G_{nm}(s, \xi)\left\{\exp(-s^2 t)\left[f_{nm}(\xi) + \int_0^t \exp(s^2 t) F_{nm}(s, \tau)\, d\tau\right]\right\} \xi^2 d\xi ds;$$
$$T_e = T_{e1} + \frac{2}{\pi} \sum_{n,m} Y_{nm}(\theta, \varphi) \int_0^\infty \frac{a_{nm} J_n(k_e r) + b_{nm}\chi_n(k_e r)}{c_{nm}^2(s) + b_{nm}^2(s)} \int_0^\infty G_{nm}(s, \xi)\left\{\exp(-s^2 t)\left[f_{nm}(\xi) + \right.\right.$$
$$\left.\left. + \int_0^t \exp(s^2\tau) F_{nm}(\xi, \tau)\, d\tau\right]\right\} \xi^2 d\xi S ds; \tag{13}$$
$$G_{nm}(s, \xi) = (G_a(s, \xi))_{nm}\ \ \xi \leqslant R;$$
$$G_{nm}(s, \xi) = (G_e(s, \xi))_{nm}\ \ \xi > R;$$
$$(G_a)_{nm} = \frac{c_{pa}\rho_a}{R^2} J_n(k_a\xi);\ f_{nm}(\xi) = q_{nm}(\xi) - (T_1)_{nm};$$
$$(G_e)_{nm} = \frac{k_e}{a_e^2}\{a_{nm}(s) J_n(k_e\xi) + b_{nm}(s)\chi_n(k_e\xi)\},$$

if the function $F_{nm}(\xi, \tau)$ is such that the orders of integration with respect to ξ and τ can be interchanged.

Since the integrand functions are even with respect to s, we can take the integral with respect to this variable over the range $(-\infty, \infty)$. Considering the integral with respect to this variable in the complex plane and choosing an integration contour in the form of the segment (−s, s) and an arc in the upper half plane, we can show that the integral over the arc (in spite of its seeming divergence) tends toward zero; the integral can then be evaluated by means of residues, for which we need to solve the equation

$$a_{nm}^2(s) + b_{nm}^2(s) = 0. \tag{14}$$

In this way we can also determine the relaxation time of the system as $\tau_p = \max\,[\mathrm{Re}(s^2) - \mathrm{Im}(s)^2]^{-1}$; the Eq. (14), however, can only be solved numerically.

To illustrate the influence of the kinetic layer and the boundary conditions (2), we supply a formula for the effective thermal conductivity in the stationary case. Averaging the thermal flow in the aerodispersed system, we have

$$\langle q_i \rangle = -(\varkappa_e)_{ij} \langle \nabla_j T_e \rangle + \int \varphi d\Omega \sum_n \int_{V_a} [(\varkappa_e)_{ij} - (\varkappa_a)_{ij}] \nabla_j T dV.$$

The function φ describes the possible states of the system. Assuming that

$$\langle q_i \rangle = \varkappa_{ij}^{\text{eff}} \langle \nabla_j T \rangle ;\ \nabla_j T = c_{ji} \langle \nabla_i T \rangle ,$$

we find, in the case of identical particles, that the effective thermal conductivity is given by

$$\varkappa_{ij}^{\text{eff}} = (\varkappa_e)_{ik} \left\{ \delta_{kj} - \frac{N}{V} \int \varphi d\Omega \int \tau_{kl} c_{lj} dV \right\} ;$$

$$\tau_{kn} = \delta_{kn} - (\varkappa_e)_{km}^{-1} (\varkappa_a)_{mn},$$

where $N/V = \gamma_N$ is the volume concentration of the particles. Assuming that the medium and the particles are isotropic, that γ_N is small, and that we can neglect the thermal interaction, we find, in this approximation,

$$\varkappa^{\text{eff}} = \varkappa_e \left\{ 1 - \gamma_N \frac{\tau \int \frac{\partial T_a}{\partial z} dV}{|(\nabla T_e)_\infty|} \right\} ;$$

$$\langle \nabla_j T \rangle = (\nabla_j T_e)_\infty ;\ \tau = 1 - (\varkappa_a / \varkappa_e).$$

With no loss of generality, we can assume that $(\nabla T_e)_\infty = (\nabla_z T_e)_\infty$. In the stationary case we obtain, carrying out the integration over the volume of the particle,

$$\varkappa^{\text{eff}} = \varkappa_e \left\{ 1 - 3\gamma_N \frac{(\lambda - 1)(1 + 2K_\tau c_q K_n^2}{1 + 2\lambda + 2\text{Kn}(K_\tau - c_q)} \right\} ;\quad \lambda = \frac{\varkappa_e}{\varkappa_a} .$$

The influence of the coefficients K_T and c_q can be characterized by the expression

$$P = \left| \frac{\varkappa^{\text{eff}} - \varkappa_e}{\varkappa_1^{\text{eff}} - \varkappa_e} \right| ,$$

where the index 1 means that in the calculations the coefficients K_T and c_q were taken into account.

According to the data in [3], $K_T = 2.2$; $c_q = 0.55$. Then with Kn = 0.3, for an air–marble dust system we obtain P = 1.9. Putting $c_q = 0$, i.e., not taking thermal slippage into account, we have P = 2.25.

NOTATION

R, particle radius; $\varkappa$, thermal conductivity; c_p, heat capacity; ρ, density; K_T, c_q, coefficients for a thermal jump and slip; l, mean path of particles of a medium; Kn, Knudsen number; $F(\mathbf{r}, t)$, function of the sources; $q(\mathbf{r})$, initial temperature distribution; T_1, stationary solution; T_2, solution regular in the infinity; $Y_{nm}(\theta, \varphi)$, spherical harmonic functions; J_n, χ_n, spherical Bessel and Neuman functions. Indices: a, particle parameters and value of the functions inside a particle; e, same in the medium; n, m, indices of the coefficients for spherical function expansions.

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